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On Gödel's "Platonism"*

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Résumé : Cet article discute des analyses de Gödel sur la réalité des objets mathématiques. Nous distinguons trois énoncés :

(i) Les mathématiques décrivent une réalité non sensible, qui existe indépendamment des actes et des dispositions de l'esprit humain.

(ii) Les théories mathématiques ne sont pas créées par l'*ego* à partir de rien.

(iii) Les mathématiques (ou quelque chose dans les mathématiques) sont indépendantes des propriétés spécifiques de l'esprit humain.

En nous appuyant sur ses archives, nous soutenons que Gödel ne peut pas adopter le platonisme fort de l'énoncé (i) après 1954. Sa position est mieux décrite par les deux énoncés, plus faibles, (ii) et (iii). Ceux-ci offrent deux sens, indépendants, de ce qu'est « l'objectivité » en mathématiques, et l'on peut très bien accepter l'un sans l'autre.

Abstract: This paper concerns Gödel's conception of the reality of mathematical objects. I distinguish three claims (i), (ii), (iii).

(i) Mathematics describes a non-sensual reality, which exists independently both of the acts and [of] the dispositions of the human mind.

(ii) Mathematical theories are not created by the *ego* out of nothing.

(iii) Mathematics (or something in mathematics) is independent of the specific properties of the human being.

I argue that Gödel cannot hold such the strong Platonism of (i) after 1954. His position is better described by the two weaker claims, (ii) and (iii). Claims (ii) and (iii) offer two different meanings for the idea of an 'objectivity' of mathematics, and philosophers can very well accept one without the other.

*. This paper uses Gödel's unpublished papers from Princeton University Library which I could visit in February 2004 thanks to a Fellowship from the Society of the Friends of Princeton University Library, and in February 2005, with the program "Preuve", from University Lille III. I wish to thank all the staff of the Rare Books and Manuscripts Department for their help during my stay in Princeton.

In philosophical literature, the name of Gödel is almost invariably associated with strong Platonism. [2] By Platonism is meant a doctrine, vaguely deriving from Plato, which holds that certain objects in mathematics have some kind of independent existence. There are various versions of Platonism, and I will discuss several ones in the course of this paper. However, Gödel himself is usually supposed to have embraced a strong, unqualified and hardly acceptable Platonism. That, I claim, comes from a misunderstanding of Gödel's writings. Relying on Gödel's unpublished papers (in particular on his drafts) to clarify the statements that he makes in his articles or in his public lectures, I will try to show that, at least after 1954, Gödel only makes weak claims on the question of Platonism. As Gödel generally does, I will use as synonymous the terms 'Platonism', 'Objectivism' and 'Realism'. Some versions of 'Platonism', in Gödel's sense, go better under the denomination of 'Objectivism'. I will come back later on on this important question of terminology. [3]

My point is not to defend Gödel's position. I first want to show that, contrary to what is commonly believed, Gödel cannot have held a strong Platonist position all along the later part of his logico-philosophical career. Moreover, during this discussion, different criteria for the objectivity of mathematics will appear and different meanings of the term 'objectivity'. Let us say, a criterion for the objectivity of mathematics is an answer to the following question: On what conditions can we admit that mathematics has an objectivity? Or what would prove that mathematics has an objectivity? A certain criterion being agreed on, the Platonist has to show that mathematics does fulfil the conditions for objectivity. The opponent can either denounce the criterion of the Platonist or argue that mathematics does not fulfil the specified conditions (he may, in fact, do both). [4] The criteria that one can find in Gödel's writings will be related to an 'epistemic constraint'. However, another question concerns the meaning itself of the objectivity of mathematics. In a nutshell, the question is: What exists independently of what? First, what does the question of existence bear on? The objects and relations, the universes described by our theories? Certain ideas

2. See for example [Parsons 1995], [Balaguer 1998], [Martin 2005]. See however [Potter 2001].

3. See in particular quotations (4), (14), (19) and section 9. There is only one exception, in a note transcribed by Wang [Wang 1996, 211]. For the rest, the three terms are used as synonymous, referring to positions described by the same criteria.

4. I take these two aspects from two-fold discussion of [Wright 1992] in [Shapiro 2007].

that are at the roots of our theories? For example, one could claim that there is an objectivity of the concept of set without supporting the objective existence of a universe of sets described by set theory in its present state. Second, should the independence that is recognised to this 'something' be referred to the human being (including the body), to the human mind as it can be analysed by introspection, or to mind, reason in general (admitting that there may exist non human rational beings such as God, angels or Martians)? For example, one could argue that mathematics is objective with respect to the human mind in the sense that we are lead to certain theories and cannot but do, say, arithmetic and set theory, although these theories depend on the structure of our brain so that mathematics is not objective with respect to our whole being nor to the mind in general.

Such questions on the meaning of the 'objectivity' of mathematics, or on the nature of an 'Objectivist' position, are, I believe, at the centre of Gödel's analysis. They determine the meaning and the strength of his Platonism. In fact, the misunderstanding of Gödel's position precisely comes from putting aside such questions. Gödel defines a 'Platonist', a 'Realist', or an 'Objectivist' position by a weak claim so that his admission of being a 'Platonist' does not imply the strong doctrine that is usually attributed to him. Aside from the reassessment of Gödel's position, my aim will be to distinguish, using Gödel's writings, different meanings of the objectivity of mathematics (different answers to a problem with two unknowns: What exists independently of what?) and to show that these different versions of 'objectivity' permit an analysis of the possible positions in philosophy of mathematics.

The bulk of this paper consists in a historical reading of Gödel's writings. Nevertheless, this reading will lead to distinctions between kinds of objectivity in mathematics. These distinctions are, I believe, of interest for contemporary philosophy of mathematics. They also permit to analyse the tradition in philosophy of mathematics and compare, with respect to precise claims, different positions. So to speak, this paper is concerned with philosophy of philosophy of mathematics, but philosophy of philosophy of mathematics may still be philosophy of mathematics.

I will start (section 1) by introducing and discussing three different claims that stem from Gödel's writings. I will then replace the two first claims in the context of Gödel's writings (section 2) and deal with two problems that arise from these claims (sections 3 and 4). In the remainder of the paper, I will explore several positions that give an objectivity to mathematics but not in the sense of a strong Platonism.

Section 5 ("Unconscious mathematics") evokes a rather unlikely option that is nevertheless hinted at in several of Gödel's notes. Section 6 concerns Brouwer's Intuitionism. It will lead to introducing the third claim discussed in section 1. Section 7 analyses Gödel's own position in 1964, which contradicts his earlier strong Platonistic claims. Section 8 deals with Husserl's Phenomenology.

As concerns the historical reading of Gödel's writings, the present paper extends the discussion of [Cassou-Noguès 2005]. I will introduce new material from Gödel's *Nachlass* and give more importance to Gödel's interpretation of the position of other philosophers, such as Husserl. Besides, Cassou-Noguès [Cassou-Noguès 2005] reviews Gödel's early arguments for his Platonism. I will, in the present paper, concentrate on what I take to be Gödel's main argument in his later writings, in order to be able to discuss with more leisure its philosophical implications. I will use the conversation with Wang (in [Wang 1996]) and the unpublished papers kept at Princeton University Library. In particular, I will make several references to the Philosophical Notebooks in C. Dawson's transcription from Gabelsberger to longhand German. This transcription is still a draft.

1 Three claims

1.1 Strong and weak Platonistic claims

It is true, there are in Gödel's writings several statements that, unambiguously, make strong Platonistic claims:

(i) "[...] Mathematics describes a non-sensual reality, which exists independently both of the acts and [of] the dispositions of the human mind."[5]

However, these strong claims are unusual. I do not know any such claims in Gödel's writings after 1954. I exclude unqualified assertions (such as in the Grandjean questionnaire, in 1975: mathematical Realism

5. The complete sentence is: (1)

"[...] the Platonistic view is the only one tenable. Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and [of] the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind." [Gödel 1951, 323]

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"was my position since 1925" [Gödel 1986–2003, IV, 447]). As already noted, the meaning of such assertions depends on what Gödel himself understands as Platonism or Realism. And I will argue that, in Gödel's terminology, after 1954, 'Platonism', as 'Objectivism' or 'Realism', refers to weak positions, defined by weak criteria. I will for example try to show that the second version (of 1964) of the paper on Cantor ("What is Cantor's continuum problem?") does not defend a strong position such as (i), but only a weaker Objectivism such as discussed below ((ii) or (ii')): Gödel in 1964 cannot hold (i). In fact, as we will see, the lecture of 1961 ("The modern development of the foundations of mathematics") seems to take an even weaker position, with a claim that frankly contradicts Gödel's earlier and later arguments for the reality of mathematical objects.

Moreover, it is clear that Gödel's beliefs fluctuate. Thus, if one could find an isolated strong claim (equivalent to (i)) after 1954, I would still argue that (i) might reflect Gödel's belief at the time the claim is made, but does not express generally the conclusion of his arguments nor what Gödel considers as a requisite for a satisfactory account of mathematics. In Gödel's view, the requisite for an account of mathematics is a weaker claim such as:

(ii) "Mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions."[6]

or

(ii') Mathematical theories are not created by the *ego* out of nothing.

Let me discuss briefly the difference between (i) and (ii), and the equivalence between (ii) and (ii'). In a nutshell, (i) excludes two possibilities, or two ranges of options, that (ii) leaves open. Let us call them possibility (A) and possibility (B).

1.2 First possibility (A) for a weak Objectivism

Claim (ii) does not necessarily give an independent reality to the objects of our mathematics such as integers or sets. It refers to 'something'

6. The complete sentence is: (2)

> "[it] seems to imply that mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions, that is to say, [it seems to imply] some form or other of Platonism or 'realism' as to the mathematical objects." [Gödel 1951, 311] (Gödel's emphasis)

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in mathematics. The objectivity of mathematics may come from 'something' out of which mathematical 'objects and facts', let us say mathematical theories, are created. Claim (ii) does not imply that mathematical theories 'describe' an ideal reality but only that they are constructed out of a reality, which they might not literally reflect. Claim (ii) only means that there is at the basis of mathematics a material, a 'something' that is not created, even though mathematical theories may be created by the human mind and may not, properly speaking, describe this 'something'.

Let us first remark that claim (ii) leaves open options that are not usually associated with 'Platonism'. For example, a traditional empiricist (in the vein of Locke or Hume) who claims that mathematical objects are created by abstraction from perceptual objects would comply with claim (ii) but not with claim (i).

But the difference, in that perspective, between claim (i) and claim (ii) may be clearer if we compare the situation in mathematics with that of sense-perception. As we will see, this analogy, between mathematical intuition and sense-perception, is crucial for Gödel's position in 1964. Let us imagine a description of an object of perception: the table on which I write is 'brown, hard, and smooth under the hand'. A 'natural realist', who believes that the objects of our perception exist as we perceive them, will say, as in claim (i), that these words describe an independent reality. Now a 'physical realist', who believes that reality is the object of physics, will refuse that these words describe an independent reality but will accept, as in claim (ii), that there is something (atoms, light waves, etc.) in the perceptual object or at the root of our perception which has an independent existence. In this first perspective, the difference between claim (i) and claim (ii) is an analogue in mathematics to the difference, with regards to perception, between 'natural' and 'physical' realism.

Gödel's position in 1964, in the second version of the paper "What is Cantor's continuum problem?" can be considered in the light of this analogy. We have—Gödel argues—a kind of abstract or "ideal sensation", on the basis of which we "form" mathematical objects. Thus, there is an ideal reality at the roots of mathematics, just as for the physical realist there is an independent reality at the roots of our perception. Nevertheless, just as the words 'brown, hard and smooth' do not describe physical reality, our mathematical theories may not describe the ideal reality.[7]

1.3 Second possibility (B) for a weak Objectivism

Claim (i) refers to the 'acts and dispositions' of the human mind, whereas claim (ii) only refers to its 'acts and decisions'. Now, in the usual sense, a 'decision' is a conscious choice. It is made with some freedom and it can be accounted for. One may ask for the reasons of such or such a decision. On the other hand, a 'disposition' refers to a faculty or an inclination, which leaves no choice and whose origin cannot necessarily be traced. There is a difference between: 'I *decide* to go to bed early this evening' (even though I do not feel sleepy, I must get up early tomorrow); 'I have a *disposition* to sleepiness in the evening' (which will eventually induce me to go to bed early even though I might rather have worked a bit longer). On one side, we have a conscious choice, on the other side, we have an inclination which we do not choose.

Now, in claim (i), Gödel declares mathematics independent of our 'dispositions', whereas, in claim (ii), mathematics only appears independent of our 'decisions'. Thus, in (ii), mathematical theories may be related to such an inclination, in the human mind, which does not involve any choices and cannot be accounted for by an introspective analysis. To take an example that is foreign to Gödel's philosophy, a Cognitivism which relates the axioms of the main mathematical theories (say, arithmetic and set theory) to the properties of the human brain appears to comply with claim (ii): for we do not freely decide our axioms but are inclined, because of the properties of our brain, to use such and such axioms. It shows that claim (ii) is a weak definition for the objectivity of mathematics.

7. With regards to this distinction, an anonymous referee suggests that Gödel could hold with respect to mathematics a third kind of realism. Consider the kind of realism hold by Locke, with respect to perception. There would be in our perception of an object properties which are entirely subjective, such as the colour—secondary qualities—and—primary qualities—properties which belong to the object in itself, such as the shape. The object in itself is acknowledged to have an independent existence. Indeed, Gödel may have hold such a realism with respect to mathematics. But it would still be too weak to comply with claim (i), for the description that we can give of the object as it is perceived is not adequate to its reality or, in other words, our sentence "brown, hard, and smooth under the hand" does not adequately describe an object which exists independently of us. One may well hold in that perspective that our representations of the object are constructed whereas the object itself has a reality of its own.

1.4 The equivalence between claims (ii) and (ii')

Thus, as we have seen, claim (ii) is weaker than claim (i). It leaves two open possibilities, (A) and (B), which are excluded in claim (i). Now the equivalence between (ii) and (ii') is partly a question of words. I will distinguish two steps.

First, Gödel considers the *ego* as the sphere of consciousness: that which produces our acts and decisions. In this manner, claim (ii), that there is something in mathematics that cannot be referred to our acts and decisions, is equivalent, in Gödel's terminology, to the claim, which we can momentarily call (ii"): There is something in mathematics which the ego has not 'created', made up or invented.

The second step, from claim (ii") to claim (ii'), seems to amount to an equivalence of the form:: $\exists x \neg P(x)$ is equivalent to $\neg(\forall x P(x))$. If there is something in mathematics which the ego has not created, then the ego has not created everything in mathematics or the ego has not created mathematics out of nothing. And conversely. In brief, saying that there is at the basis of mathematics a material which the ego has not created is equivalent to saying that the ego has not created mathematics out of nothing.

Thus, claim (ii) is equivalent to claim (ii"), which is equivalent to claim (ii'). We will see different texts where Gödel uses claims (ii) and (ii') as equivalent (in particular quotation (4) below). Let us stress that claim (ii'), the ego has not created mathematics out of nothing, may only mean that there is something in mathematics, which exists objectively and independently of our mental acts and decisions, and does not necessarily imply that mathematical objects themselves exist objectively and independently of our mental acts and decisions (see earlier possibility (A)).

1.5 A third claim

In claim (ii), Gödel only denies that mathematical theories are created out of nothing and by the *ego*, that is in a conscious activity that can be reflected upon and analysed. As [Cassou-Noguès 2005], I will argue that claim (ii) is Gödel's only requisite with respect to 'Platonism'. It is a fact that Gödel does consider as examples of 'Platonism' positions that comply with (ii) but not (i). Already, in the second quotation, Gödel describes as 'Platonism' a position that is weaker than the one defined in the first quotation.

However, precisely because claim (ii) opens up a wide range of possible positions (such as a Cognitivism or, as we will see, an Intuitionism in the style of Brouwer's), it does suffice to define Gödel's own position. To reach a position that comes close to Gödel's own, one must add another claim, which, in Gödel's view, is not directly related to the question of Platonism:

(iii) Mathematics (or something in mathematics) is independent of the specific properties of the human being.

An account of mathematics should make it clear that mathematics, or something in mathematics, does not depend on properties that would be characteristic of the human being and would not belong to any other rational being. This third claim excludes positions like a Cognitivism or, as we will see, Intuitionism.

Claim (iii) offers a specific problem. In claim (ii), the question is whether mathematics can be considered as created in, or let us say invented by the ego, the mind as it can be analysed by introspection. In claim (iii), the question is: to what extent does mathematics depend on our humanity? The two questions have a different orientation. It is perfectly possible (we will see an example with Husserl) to argue that mathematics is created by the *ego* but that no essential feature of this creation is related to a specifically human nature.

In Gödel's writings, the terms 'Objectivism', 'Platonism' or 'Realism' are applied to any position complying with claim (ii). Gödel does not consider Cognitivism but, since Brouwer's account of the origin of numbers seems to comply with claim (ii), Gödel explicitly considers intuitionism as a form of 'Objectivism'. However, in wider perspective, one may consider claim (iii) as an aspect of what is called the 'objectivity' of mathematics in contemporary philosophy. If one is reticent to consider Cognitivism as being on the side of the objectivity of mathematics, it is because Cognitivism does not fulfil the conditions of (iii) even though it may fulfil those of (ii). Claims (ii) and (iii) correspond to two different meanings of the idea of 'objectivity'.

Now, I must admit that I have not found claim (iii) explicitly formulated in Gödel's writings (one must remember however that only about one half of the Philosophical notebooks have yet be transcribed from Gabelsberger). I will argue nevertheless that (iii) is an essential feature of Gödel's philosophy of mathematics (see sections 7 and 8). In fact, Gödel's position can be summarised as the conjunction of (ii) and (iii). The addition of (iii) to (ii) does certainly narrow the field of possibilities but these two claims do not contradict each other nor is their conjunction equivalent to (i).

Let us compare again (as in 1.2.) the situation in mathematics with that of perception. I describe the table on which I write as ‘brown, hard and smooth’. If I am a physical realist, I will say that this description is not adequate to the reality, which is made of atoms and light waves, but that there is something in this ‘brown, hard and smooth’ object, or at the root of this object, which has an objective reality. I might also add, as in claim (iii), that, therefore, there is, in this ‘brown, hard and smooth’ object, or at the root of this object, something which is independent of the specific properties of the human being. In the same way, if, as Gödel believes in 1964, we “form” mathematical objects on the basis of an ideal sensation, we may refuse claim (i) but accept claims (ii) and (iii).

I will discuss at length the relations between (i), (ii), (ii’) and (iii) on the example of a number of philosophers, using Gödel’s own interpretations of their writings. I will argue that Gödel’s position in 1964 complies with (ii), (iii) but not with (i), that Brouwer, in Gödel’s view, complies with (ii) but not with (iii), and that Husserl might comply with (iii) but not with (ii). I will also argue that Gödel changed his position at least twice. I will distinguish three steps: before 1951, when he holds a strong Platonism such as described by (i); 1961, when can only be attributed to him a position even weaker than (ii) or (ii’); 1964 and after, when he seems to settle on the weak Platonism described by (ii) or (ii’). I see no evidence of further changes after 1964. Now one question is what motivates these changes. I will try (at the end of section 8) to give a brief summary of Gödel’s moves but I simply have no answer as to their motivations. [8]

2 The argument for claim (ii)

The central claim of Gödel’s later writings is not that mathematical objects, such as integers or sets, have an independent reality. It is rather that mathematical objects have not been created ‘by the *ego*’ and ‘out

8. Above all, my aim is to show that Gödel does no longer believe in (i) after 1961, contrary to what is commonly believed. Now an anonymous referee for this paper suggested that Gödel may never have hold (i). So let us underline that in 1951 Gödel does explicitly define Platonism as (i) and considers this Platonism as the “only tenable position” (see above quotation (1) footnote 5)).

of nothing'. This idea of a 'creation out of nothing' first appears in a lecture of 1951.

It is related to a disjunction on which Gödel bases much of his latter philosophy. Indeed, Gödel relies on his incompleteness theorem, as it can be reformulated in terms of Turing-machines, to establish that

> (3) either the human mind is not a Turing machine or there exist arithmetical propositions that will remain absolutely undecidable.

Both alternatives go against what Gödel calls materialism [Gödel 1951, 310]. Since, according to Gödel, the brain is a Turing-machine, the first alternative shows that there is in the mind something that has no correlate in the brain and, therefore, that exists independently of the brain or any other material object. But the second alternative also leads to the conclusion that there are such non-material entities. The existence of absolutely undecidable propositions would imply, according to Gödel, that there are some data at the basis of mathematics, which we have not created and which simply cannot be material. More precisely, mathematics is not 'our own creation':

> (4) "For the creator necessarily knows all the properties of his creatures, because they can't have any others except those he has given to them. [... Therefore,] mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions, that is to say, [it seems to imply] some form or other of Platonism or 'realism' as to the mathematical objects. [...] One might object that the constructor needs not necessarily know *every* property of what he constructs. For example, we build machines and still cannot predict their behavior in every detail. But this objection is very poor. For we don't create the machines out of nothing, but build them out of some given material. If the situation were similar in mathematics, then this material or basis for our constructions would be something objective and would force some realistic viewpoint upon us even if certain other ingredients of mathematics were our own creation. The same would be true if in our creations we were to use some instrument in us but different from our *ego* (such as 'reason' interpreted as something like a thinking machine)" . [Gödel 1951, 312]

The last but one sentence was first written as:

> (4') "Similarly, of course, if mathematics were not created by *our* acts (i. e. conscious acts of our *ego*) but by some entity in us called reason, mathematical facts would be something objective namely properties of this mind, which we could in no way determine by our choices." [Gödel s. d. a., box 8b, folder 93, item 040294]

First, one can see in quotation (4) the equivalence between claim (ii) and claim (ii'). What I earlier called claim (ii) is stated as the second sentence of quotation (4). The reason, according to Gödel, for claim (ii) is that "the creator necessarily knows all the properties of his creatures" (in the first sentence of quotation (4)) whereas Gödel has, in this passage, admitted the existence of absolutely undecidable propositions. Then Gödel distinguishes two cases in which it would be false that "the creator necessarily knows all the properties of his creatures": (A) if the creation was based "on some given material" (in the fifth sentence) and (B) if "in our creation we were to use some instrument in us but different from our *ego*" (in the seventh sentence or in quotation (4')). What I earlier called claim (ii') excludes exactly these two cases: mathematical objects have not been created by the ego out of nothing. Thus claim (ii') implies claim (ii). Conversely, it is clear that if the ego has created mathematical objects out of nothing, one cannot hold claim (ii), that mathematical objects (or something in them) exist objectively and independently of our mental acts and decisions. Thus, claim (ii) and claim (ii') are equivalent.

Second, one sees in quotation (4) that Gödel qualifies as "Platonism" or as "realism" a position that complies with claims (ii) and (ii'), but not necessarily with a strong claim such as (i). This will be important when we come to the case of Intuitionism (in section 5).

Now, to sum up the point of quotation (4), Gödel here argues that we cannot have invented mathematical theories, in our *ego* and out of nothing. We cannot have created mathematical objects in this way, for, if we had, we would know all that we have put in them, we would know all their properties and there would be no undecidable propositions. However, mathematical objects as such need not have an independent existence. Gödel leaves open two possibilities: (A) that mathematical objects have been created on the basis of something else, so that mathematics has an objectivity even though its terms, such as integers, sets, are human creations; (B) that mathematical objects are produced in a part of our mind, a 'reason', which is different from our *ego* and whose workings remain absolutely unconscious. Gödel always maintains the clause that, when he denies that mathematics has been created by us, he understands the term 'creation' in a specific sense: creation out of nothing.

> (5) "Gödel emphasises that to create means to make out of nothing. That is why in creating mathematical objects, we would give them all their properties. If to create were just making something out of something else, then the situation would be different."[9]

Gödel's argument immediately raises several problems. First, in quotation (4), the argument appears in relation to dilemma (3). It then relies on the hypothesis that there exist absolutely undecidable propositions, propositions that the human mind will never be able to prove nor refute. But Gödel himself does not believe in the existence of absolutely undecidable propositions. He is convinced that the human mind can solve any problem that it can formulate, either prove or refute any proposition that it can understand: "For clear questions posed by reason, reason can find clear answers" [Gödel 1961, 381].[10]

However, Gödel's argument can also be seen outside the preceding dilemma and without the hypothesis of absolutely undecidable propositions. Indeed, the following quotation gives a weaker criterion for the objectivity of mathematics. It is related to an introspective, or phenomenological, reflection: we have not invented mathematical theories out of nothing, for, if we had, we would be able to learn all the properties of their objects (prove all theorems of arithmetic) by reflecting on our acts in creating these theories.

> (6) "[...] if man has created mathematical objects in the sense of having made (i. e. imagined) them out of nothing (not having built them out of something given like we build a car), he must be able to know all their properties. For since

9. In a text from Wang corrected by Gödel [Gödel s. d. a., box 3c, folder 210, item 013195].

10. Gödel here takes up Hilbert's motto: there is no *ignorabimus* in mathematics. Of course, after the theorem of 1931, this does not mean that mathematical theories should be decidable but only that the undecidable propositions of a mathematical theory should be decidable in a stronger theory on the basis of evident axioms. Concerning dilemma (3), Gödel believes both that the mind is not a Turing machine and that there exists something in mathematics which the *ego* has not created, but he only succeeds in proving a disjunction, instead of this conjunction.

> nothing was there before, everything in the thing created must have been put into it by the creator and therefore the creator either must know of it [crossed out: if the creator was conscious or the creation such that it can be made conscious] or must be able to learn it simply by raising his creative activity from the subconscious to the conscious level (if he could not do that, it would not be his creation but a creation by something in him inaccessible to him)." [Gödel s. d. a., box 8c, folder 117, item 040396]

We will come back to this text while discussing the relationship between Gödel and Husserl (section 8). But, here, the objectivity of mathematics, the existence of something in mathematics which the *ego* has not created, does no longer comes from the assumption of absolutely undecidable propositions but from the uselessness or, at least, the incompleteness of a phenomenological, introspective reflection. We may reflect on our thought processes or, to use a phenomenological terminology, our 'life-experiences' when we think about numbers, but we cannot in this way discover all the properties of numbers, it follows that there is something in number theory which was not created by our thinking. Even more clearly, the next quotation relates the objectivity of mathematics to our actual ignorance. The simple fact that mathematical objects have properties that we still do not know implies their independence from the *ego*:

> (7) "It seems to me that the philosophical conclusion drawn under the second alternative [the existence of absolutely undecidable propositions in dilemma (3) above], in particular conceptual realism (Platonism) are supported by modern developments in the foundations of mathematics also, irrespectively of which alternative holds. The main arguments pointing in this direction seems to me [to be] the following. First of all, if mathematics were our free creation, ignorance as to the objects we created, it is true, might still occur, but only through the lack of a clear realisation as to what we have created (or, perhaps, due to the practical difficulty of too complicated computations). Therefore it would have to disappear (at least in principle, although perhaps not in practice) as soon as we attain perfect clearness. However, modern mathematics has accomplished an insurmountable degree of

> exactness, but this has helped practically nothing for the solution of mathematical problems." [Gödel 1951, 314]

It is clear in this text that our actual ignorance, the fact that there are mathematical propositions which are undecidable in the current state of our knowledge and despite the 'perfect clearness' achieved in the foundations of mathematics, implies the objectivity of mathematics.

Quotations (4), (6) and (7) offer three different criteria for the objectivity of mathematics in the sense of claim (ii): that there is something in mathematics which we have not created in an activity that can be reflected upon and completely analysed. In quotation (4), the criterion is the existence of absolutely undecidable propositions. In quotation (6), it is the mathematical sterility of a reflection on our thought processes or, at least, the fact that such a reflection does not suffice for solving open problems. In quotation (7), the criterion is our actual ignorance.

3 An objection

Now, Gödel realises that his argument, in quotation (7), can be the target of an objection. As an exercise of logic, I devise axioms in the predicate calculus for a certain domain of objects with certain relations. I simply write down axioms specifying the properties of these objects and their relations. I choose them arbitrarily, with no specific idea in mind. And I start investigating the theorems following from these axioms. It seems that I have created a mathematical theory from nothing and in what appears to be a conscious act. These objects have properties which can be deduced from the axioms but which I do not yet know and which I will not discover simply by an introspective analysis of my activity (as in (6)). At this point, I do not even know if my axioms are consistent. [11]

Gödel does give an answer to this objection. The text follows quotation (7):

> (8) "Secondly, the activity of the mathematician shows very little of the freedom a creator should enjoy. Even if, for example, the axioms about integers were a free invention, still it must be admitted that the mathematician, after he has imagined the first few properties of his objects, is at an end with his creative ability, and he is not in a position also to

11. This objection was raised by M. Detlefsen in personal conversation.

> create the validity of the theorems at his will. If anything like creation exists at all in mathematics, then what any theorem does is exactly to restrict the freedom of creation. That, however, which restricts it must evidently exist independently of the creation." [Gödel 1951, 314]

If one admits that the axioms of arithmetic are a free creation, or when one freely imagines new axioms, theorems must still be deduced from these axioms according to the rules of the calculus, predicate calculus, [12] and new axioms can only be introduced if they do not contradict the axioms already stated. Now Gödel seems to argue that these rules restrict the freedom of the creator and must therefore have a reality, which is independent of the creator. In the case when one arbitrarily devises a system of axioms, and imagines a new theory, this creation presupposes the logic that governs the theory. It seems that this logic is the 'something' out of which the theory is constructed and that it has a reality of its own. Now, the rules of predicate calculus can be formulated as schemata of inference. Gödel's answer to the objection only makes sense if we admit that these schemata do not by themselves determine what the mathematician has to do. For, if not, the simple fact of writing the schemata would account for the logic of the theory. Gödel's answer then means that, in order to apply the rules of the calculus, the mathematician needs some intuition, some capacity, some inclination, which is not by itself formulated in the schemata. It raises a problem comparable to the problems of rules in Wittgenstein.

But let us put it in another way, which is closer to Gödel's writings. Given a system of axioms in predicate calculus, a Turing machine can deduce all the theorems of the theory. In fact, giving a system of axioms is equivalent to giving a table of instruction for a Turing machine. Gödel identifies then formal systems and Turing machines. A formal system, according to Gödel, *is* a Turing machine. [13] Now, in quotation (8), Gödel seems to argue that a mathematical theory, considered as a conjunction of axioms, freely chosen, at least presupposes a logic or the ability to deduce formulas according to rules. One could say that the

12. Let us consider only the case when the axioms are formulated in the predicate calculus. If the axioms were formulated in a second order calculus, they would presuppose anyway the notion of subset. The objects could not be said to be created "out of nothing".

13. "A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas." [Gödel 1934, in Gödel 1986-2003, I, 370]

position of such a theory, which amounts to devising a Turing machine, presupposes the concept itself of a Turing machine. Indeed, the concept of Turing machine is, according to Gödel, an existing concept, which was implicit in mathematical logic before Turing and that Turing only made explicit. Turing did not create a concept but only made us see it more clearly. [14] Eventually, Gödel's answer to our objection would be that mathematical creations at least presuppose one existing concept, that of a Turing machine.

This point may have already been hinted at in the first version of the argument (in quotation (4)). Gödel remarks, "we build machines and still cannot predict their behaviour". But the objection that we were discussing amounts to saying that one may imagine new axioms (that is, imagine a new Turing machine) and yet not know from the start whether the axioms are consistent (that is, not be able to predict the behaviour of the machine). But, as Gödel adds, these machines are built, or imagined, from something else, which, in the case of Turing machines, can only be the concept itself of a Turing machine.

4 The scope of Gödel's argument

A second problem concerns the scope of Gödel's argument. In a nutshell, an object that has properties which we do not immediately know and which we cannot choose arbitrarily, needs not have an independent reality as such but must either come from an unconscious part of our mind or be constructed out of something else. That argument could apply to any object, not only mathematical objects, but also perceptual and fictional objects. Perceptual objects obviously have properties, which we do not know and which cannot be chosen arbitrarily. The same could be argued of fictional objects, like characters in a novel. The

14. "It is well known that A. M. Turing has given an elaborate definition of the concept of a *mechanically* computable function of natural numbers. This definition most certainly was not superfluous. However, if the term 'mechanically computable' had not a clear, although unanalysed, meaning before, the question as to whether Turing's definition is adequate would be meaningless, while it undoubtedly has an affirmative answer." [Gödel 1972, in Gödel 1986-2003, II, 275, note 5]

"The sharp concept is there all along, only we did not perceive it clearly at first. [...] We had not perceived the sharp concept of mechanical procedures before Turing, who brought us to the right perspective. And then we do perceive clearly the sharp concept"; conversation with Gödel quoted in [Wang 1996, 232, 235], [Wang 1974, 84].

writer does not seem to invent his or her characters step by step.[15] They appear as a whole and the writer must let them live so to speak, listen to them, before he or she can describe their features and learn their story. Fictional characters seem to fall under the conditions of Gödel's argument. Of course, one would be tempted to relate fictional characters to the unconscious of the writer. The writer would create his character unconsciously, or to use Gödel's word, in a part of his mind different from his *ego* and over which his *ego* has no control. That would be why they have properties that the writer himself does not know they have. At least, it is more tempting to relate fictional objects to the unconscious than mathematical objects. However, Gödel himself always chooses the realistic side in his argument.

> (9) "Fictional characters are empirical. In contrast, the concept of set, for instance, is not obtained by abstraction from experience"; conversation with Gödel quoted in [Wang 1996, 138]

Fictional characters, according to Gödel, would then be constructed out of something else. They would be abstracted from their model, contrarily to mathematical objects, which would have no such model in perceptual experience. One might raise objections against this position. But, at least, it is consistent with Gödel's argument on the reality of objects.

There is an earlier note on fictional characters. It was written in 1942. Gödel did not have his argument, which seems to appear in 1951.

> (10) "With regard to the sentences of a fiction *(Sätzen der Dichtung)*, one can also ask whether they are true or false. So the writer creates a peculiar reality. But, because of the human imperfection of the writer (in contrast with the creator of the real world), that question might not always have an answer (for example, he did not think of that or he contradicted himself)." [Gödel s. d. b., XI, 19]

This second note is rather ambiguous. On one hand, fictional objects are set in a parallel world, where sentences concerning them could be true or false. Gödel (vaguely) anticipates the theory developed by D. K. Lewis. On the other hand, this world is said to be 'created' by the

15. Among numerous possible examples, see [Giono 1961], Stevenson's "Chapter on Dreams" in [Stevenson 1892].

writer, and consciously created (since the writer must *think* of this or that). Thus fictional characters, as elements of this parallel world, do not have an independent reality or any kind of objectivity. This early note is not consistent with Gödel's argument on the reality of objects.

5 Possibility (B): dream mathematics

I will for the rest of this paper concentrate on the question of mathematical objects. As we have seen, when Gödel argues that mathematical objects are not 'created', he means 'created' by the *ego* (that is, in a conscious activity) and out of nothing. That leaves open two possibilities: (A) that our objects are created on the basis of something else (as Gödel seems to believe even in the case of fictional characters), or (B) that our objects are the product of an unconscious part of our mind (as one would be tempted to believe in the case of fictional characters). I will start with this possibility (B).

That option would turn mathematics into something like a dream, a story that is unconsciously produced and of which we do not know the end, nor the meaning, though we are making it up. However, Gödel does not dismiss at first hand possibility (B). In fact, there are, in Gödel's papers, several items relating to this 'unconscious' reason. They all oppose the 'ego' as the domain of conscious acts, and a 'reason' which would be "some instrument in us but different from our *ego*" (see (2)), "something in him inaccessible to him" (see (5)), some entity "in contact with our *ego* but to which our *ego* has no access", "an entity different from it but in contact with it". And, in that sense:

> (11) "In mathematics, [the] question is to find out what we have perhaps unconsciously created." [Gödel s. d. b., box 8c, folder 117, item 040403]

And, after a similar remark, in another note,

> (11') "The method for the foundation of knowledge is then psychoanalysis." [Gödel s. d. a., V, 344]

However, I do not know any item in Gödel's papers that would clearly explain what this 'reason' could be, how mathematical objects could be related to such an unconscious part of our mind. Still, there may be two ways of seeing this reason. First (as already discussed in [Cassou-Noguès 2005]), Gödel asserts in several places that science relies on a conceptual apparatus that is constituted during childhood. In a letter to C. Reid, Gödel speaks of:

> (12) "the conceptual outfit, which, in our culture, we acquire in about the first 15 years of our life and which is in no way enlarged but only applied in a more and more involved fashion by science today." [Gödel s. d. a., box 1c, folder 137, item 011853]

Now, these concepts, acquired during childhood, could then become (to use Husserl's vocabulary) 'sediments'. Our concepts are related to certain activities (both mental activities and practices in the real world). The process of sedimentation (which Husserl analyses with respect to the "Origin of geometry" in [Husserl 1936]) describes the way an activity that is at the origin of a concept obliterates itself after the concept is constituted. The concept remains in use but the activity that has produced the concept falls into oblivion, so that the constitution of the concept cannot be re-enacted, and the concept itself (its meaning, its scope, its origin) now appears as problematic. Such a process could apply here. If we form the basic concepts of our science, including mathematical concepts, during our childhood, their 'sedimentation' would mean that the adult cannot analyse their origin. Our concepts would appear then as unconscious products. Now, Gödel relates such mathematical concepts as the concept of set to the concept of object, the concept of a 'thing'. In several notes, Gödel seems to assert that the concept of set could be derived from the concept of object, as it is used in daily life or natural language.[16] In that perspective, the child would form the concept of object, and the adult could derive from it the concept of set, without being able to analyse its origin, without being able to re-enact the processes that have given him this 'something' out of which he now builds his mathematical theories. I will give later on another quotation on Husserl that also goes in this direction.

But there is a second way to look at our mathematical unconscious. It is more metaphysical. In an earlier note, of 1946, Gödel mentions the possibility that we see mathematical truths in God's mind. In that

16. For example, "in the generation of the idea of *one* object out of its various aspects, if we abstract from the interrelations of the aspects, the one object generated would be the set of which the aspects are constituents provided we thought of these aspects as objects." [Gödel s. d. a., Box 3c, folder 207, item 016167]. It is a text from Wang, but corrected by Gödel and entitled "Quotations from Gödel". See also [Cassou-Noguès 2005].

perspective, this reason in us (over which we have no control) could be conceived as a part of God's mind, linked to our *ego* or to which our *ego* would be in contact. We would then take mathematical truths, or some of them that we use as axioms, in God's mind and without being able to understand its workings or the way 'He' arrived at these truths.[17]

I do not argue that either of these two perspectives expresses what Gödel believes.[18] They simply represent two ways of understanding this mathematical unconscious, which Gödel mentions, consistently with his writings. Another way, which would not be congenial to Gödel's perspective, would be, as already noted, to relate mathematics to the brain—the axioms used in the main theories to the properties of the brain. Mathematics would then be objective in the sense of claim (ii).

6 Brouwer and Dedekind: towards 'objectivity' in the sense of claim (iii)

A surprising example of the weak Objectivism expressed in (ii) is, in Gödel's view, Brouwer's account of the origin of arithmetic. Brouwer relates the origin of numbers to the passing of time in a (mythical) consciousness. At the beginning, the consciousness is filled with a unique sensation. But, with the passing of time, this sensation divides itself in two, a sensation that is now past and a new sensation that is present. This later sensation again divides itself in two, and so on. One simply has to ignore the actual content of these sensations to obtain the series of natural numbers.

Gödel sees in this account another example of what he means by 'creation out of something else'. Indeed, numbers, in Brouwer's account, are not created freely; we do not choose their properties. Numbers are

17. Gödel remarks that the right, or the objective, formulation of a mathematical proposition is "the one that is realised in God's understanding": "It is seen in God" [Gödel s. d. b., XIV, 7–8]. One finds a similar remark: "Ideas and eternal truths are pieces of the divine substance. It does not follow that God has created them (since God does not create itself) but they make the essence *(das Wesen)* of God" [Gödel s. d. b., XI, 31].

18. An anonymous referee for this paper suggests that the second position—we see mathematical truths in God's mind—is the right one and describes Gödel's Platonism. I don't think that, even though Gödel does mention this possibility, there are many textual evidences for it. Moreover, since this position is metaphysically expensive, to say the least, it would still be worth trying to find in Gödel's texts hints at more plausible positions on mathematical objectivity.

created from or according to the passing of time, which, in Brouwer's text, appears as a phenomenon over which we have no power. Numbers have then objectivity. Or they can be considered as created from something else that has objectivity. That is enough to comply with Gödel's requirement:

> (14) "'The intuition of one-twoness', according to Brouwer, 'creates not only the numbers one and two, but also all finite ordinal numbers, etc.' Here the meaning is to make something out of something else, it is construction according to certain principles. If we create mathematical objects in this sense, it does not mean that there are no [here H. Wang puts a question mark which Gödel crosses out] mathematical objects not created by us. *Creation in this sense does not exclude Objectivism.*"[19]

Brouwer's thesis that natural numbers are the creations of a Consciousness does not in itself contradict Gödel's notion of objectivity. Brouwer complies with Gödel's weak claims for 'Objectivism' or, indeed, 'Platonism'. It is true that Gödel does not—and it would be awkward to—qualify Intuitionism as 'Platonism'. Gödel only uses the term 'Objectivism'. However, we have seen earlier (in quotation (4) in particular) and will see again (in quotation (19) in particular) that Gödel does qualify as 'Platonisms' positions complying only with the same weak claims, (ii) or (ii'): that the ego has not created mathematical objects out of nothing. The same claims define, in Gödel's terminology 'Objectivism', which he uses very rarely, and 'Platonism'. With the exception of a note transcribed by Wang [Wang 1996, 211], the two terms always seem to be synonymous.

That Brouwer complies with Gödel's claims for the objectivity of mathematics shows the weakness of these claims. Indeed, it comes from

19. It is a text from Wang which he describes as "a few more pages of my notes of [Gödel's] sayings". It is heavily corrected by Gödel [Gödel s. d. a., box 3c, folder 209, item 013183]. Also, in another note written by H. Wang corrected by Gödel: "Both Brouwer and Riemann speak of creating mathematical objects in our mind. But they do not mean or at least cannot cons[istently] mean creations [Gödel adds: out of nothing]. Rather they must mean creation [Gödel adds: out of some given element and] according to principles [Gödel crosses out Wang's text and adds: which these elements permit]. The idea is analogous to physical construction, where we can combine given material to make new things [Gödel adds: but we can combine them only in ways permitted by these things (e.g. we cannot combine the hardness of the diamond and the chemical composition of the wood)]" (in box 20). See also [Wang 1996, 225].

the fact that Gödel only refuses the idea of a creation by the *ego* and *out of nothing.* It is interesting to compare this use of the idea of creation in mathematics to that of Dedekind in *The nature and meaning of numbers.* In the preface, Dedekind asserts that "numbers are free creations of the human mind" [Dedekind 1888, 31].[20] However, by this, Dedekind does not mean that numbers would be freely created by the mind out of nothing. Creating, for Dedekind, rather means that starting from a certain set (a simply infinite system) one can ignore the particular content of its elements and retain only the properties one does need for doing mathematics. The numbers are the result of this abstraction. But, since one chooses the properties that one retains, and since numbers, after this abstraction, are true objects, different from the ones which they have abstracted from, the mathematician is justified in seeing numbers as her or his creations: "With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind" [Dedekind 1888, 68].

It is because of his peculiar use of the idea of creation that Gödel may object to Brouwer that, in his own doctrine, numbers are not freely created: they are created out of something else.[21] Brouwer's account, interpreted in this way, then seems to comply with Gödel's requirement for the objectivity of mathematics. However, there still must be something in Brouwer's position that Gödel rejects. The divergence between Brouwer and Gödel (concerning what mathematical theories should be, what kind of axioms and inferences may be used) is too wide to be simply a matter of personal choice, as if Brouwer's account could be acceptable to Gödel but simply less appealing than a clear-cut Platonism. The question, therefore, is to understand exactly what in

20. This point was suggested to me by M. Detlefsen.

21. Indeed, this reading of Brouwer is a criticism at least of the letter of the intuitionist philosophy. As M. Detlefsen shows, Brouwer's requirement that mathematical objects be constructed in intuition, and that an exhibition of the objects be always possible, is related to the idea (stemming in Brouwer's case from German idealism) that subjective acts are transparent to themselves, so that the subject can always analyse what he has himself constructed. Brouwer's requirement "is to be understood in terms of the supposedly special knowledge that a freely creative agent has of her free creations. The idea behind it is roughly that *our epistemic position with respect to that which we create is better than our epistemological position with respect to that which we do not create*" [Detlefsen 1998, 319; M. Detlefsen's emphasis]. M. Detlefsen clearly uses Gödel's vocabulary. However, what Gödel argues against Brouwer, is that his account of arithmetic starts with the intuition of time, which itself is not created by the subject and which therefore remains as an opacity, blurring the clear vision that the creator should have of his creations.

Brouwer's position Gödel eventually rejects. It is not in itself the thesis of a creation of mathematical objects, which, as we just have seen, may be acceptable to Gödel.

There may be a difference concerning the nature of mathematical intuition but I think that Gödel's main divergence with Brouwer is related to the status of time. In Brouwer's philosophy, time may be a phenomenon that belongs to any consciousness, any mind. However, in Gödel's view, time is a character of our mind that does not belong to any mind:

> (15) "For a being which had no sensibility at all (i. e. no contact through sensations with reality) but only 'pure understanding', no time at all would exist." [Gödel 1986–2003, III, 427]

Or, with another specification, in an earlier paper:

> (16) "For a being with no sensibility at all [crossed out: i. e. not imbedded in the world with our sense organs] (i. e. which would have no body in the material world) but were to consider it only from the outside (by 'pure understanding') no time would exist [...]. It must be ass[umed] in addition that the understanding of this being were so perfect that it does not need marks on paper (or memory pictures in the brain) (which as material processes are only possible in space and time) as crutches but would penetrate all conceptual rel[ations] in one glance." [Gödel s. d. a., box 9a, folder 131, item 040418]

In Gödel's view, this being or, maybe, these beings, of which God is an example, should be able to do mathematics, and arithmetic. But, according to Brouwer's account, they would not, for arithmetic appears to depend on time, whereas they do not know time. What Gödel refuses in Brouwer's account is not in itself the thesis of a creation of numbers but the fact that arithmetic is related to time, which is a specific property of the human mind.

The thesis that arithmetic should not be related to time appears explicitly with reference to Kant. As is well known, Kant distinguishes in the *Critique of Pure Reason,* between analytic judgments, where the predicate is contained in the concept of subject, and synthetic judgments, where the predicate is not contained in the concept of the subject, and therefore, where the predication relies on intuition. In Kant's view, mathematical propositions are synthetical. Geometrical propositions rely on the intuition of space. Thus, in order to prove that the sum of the angles of a triangle is equal to a flat angle, the geometer draws a figure, a triangle and a straight line going through one summit and parallel to the opposite side of the triangle. The geometer literally sees the theorem on the figure. On the other hand, arithmetical propositions rely on the intuition of time. To verify an equation, such as 5+7=12, one must count, count five apples, then seven apples, which makes twelve apples. Now, to count a series of objects, one must be able to notice them one *after* the other. In that, arithmetic is related to the time structure of our internal life. Geometry and arithmetic, in Kant's view, are *a priori* because they refer to these two forms of intuition that are innate in the subject: space as the form of external intuition, time as the form of internal intuition.

In an early paper, "Intuitionism and formalism", from 1912, Brouwer refers to Kant's theory of arithmetic. According to Brouwer, non Euclidean geometries refute the Kantian thesis that space, Euclidean Space, is the *a priori* form of external intuition. We can prove, in the same way, propositions that refer to objects in the Euclidean space and propositions that refer to objects in a non Euclidean Space. Therefore, if these proofs are based on a construction in intuition, the form of this intuition cannot be the Euclidean space. The first thesis, that space is an *a priori* structure, must be abandoned. However, Brouwer takes up the second thesis, that time is the *a priori* form of internal intuition, i. e. the fixed structure of our mental life, and that arithmetic is founded on the intuition of time.

On the other hand, Gödel explicitly rejects this Kantian thesis. He refers to an earlier essay where Kant had not yet developed his theory of arithmetic. Gödel notes:

> "This writing is interesting also for the reason that it avoids the faulty analogy: arithmetic/time and geometry/space but instead holds that the concept of time gives rise to cinematic while the concept of number is considered to belong to the sphere of abstract thinking and to require pure intuition (of either space or time) only for its *actuatio concreto*." [Gödel s. d. a., box 8b, Folder 95, item 040295, draft for Gödel 1951]

As show quotations (16) and (17), the refusal to relate arithmetic to time comes from the fact that, in Gödel's view, there exist rational beings outside time, such as God, who should be able to do arithmetic or, more generally, some mathematics comparable to ours. Therefore, there must be at least something in mathematic which is independent of this specific property of human minds: being in time.

This discussion on Brouwer's intuitionism leads us to introduce a new claim (earlier claim (iii)). This claim gives a new meaning to the objectivity of mathematics. However, though it is at the root of Gödel's rejection of Brouwer's account of arithmetic, this third claim does not seem to appear explicitly in Gödel's writings. Gödel qualifies as 'Objectivism' positions such as Brouwer's complying with claim (ii). As we have seen in quotation (14), Gödel explicitly acknowledges that Brouwer's account illustrates his idea of 'Objectivism', since numbers appear to be created out of something over which the *ego* has no control. Gödel's own position concerning the nature of mathematics seems to be determined by the conjunction of claim (ii) and claim (iii). But, in his terminology, claim (iii) is not related to the question of 'objectivity', which is solely determined by claim (ii).

7 Gödel in 1964

Let us look again at claims (i), (ii) and (ii'). Brouwer's account, according to Gödel, complies with claims (ii) and (ii'). But it can in no way comply with claim (i). That shows (and we will see again later on) that claim (ii) is sufficient to characterise an 'Objectivist' position. Claims like (i) may have expressed Gödel's believes in 1951 (or until 1954).[22] But they do not express the conclusion of his argument (see quotation (2)). And it seems that Gödel himself will abandon such a position as defined by (i), in favor of a weaker position defined by claims (ii) and (iii). I will take the example of the supplement of 1964 to the paper "What is Cantor's Continuum Problem?"

At first sight, Gödel does seem to make strong Platonistic claims. The question of mathematical objects is considered as analogous to the question of sense objects, the objects of perception:

22. A claim comparable to (i) still appears in a letter to G. Gunther from 1954. Gödel writes: "[...] I say that one can (or should) develop a theory of classes as objectively existing entities. [...] They seem rather to form a second plane of reality, which confronts us just as objectively and independently of our thinking as nature" [Gödel 1986–2003, IV, 504–505].

> (17) "The question of the objective existence of the objects of mathematical intuition [...] is an exact replica of the question of the objective existence of the outer word."
> "I don't see any reason why we should have less confidence in this kind of perception, i. e. in mathematical intuition, than in sense-perception." [Gödel 1964, 268]

Mathematical objects must then have the same reality as sense objects. But, precisely, Gödel denies that sense objects as they appear to us have an independent reality. Sense objects are constituted on the basis of sensations. We then contribute, by certain mental processes, to shaping our objects from the data that we receive from the outside. The same goes for mathematical objects:

> (18) "It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else, which *is* immediately given. Only this something else here is *not,* or not primarily, the sensations." [Gödel 1964, 268]

We have a mathematical intuition. That, for Gödel, is "a mere psychological fact" [Gödel 1964, 268]. It does not imply that the objects of our theory have an independent reality. The objects are already informed by the subject, just as the objects of perception. Mathematical objects are merely constituted by the subject on the basis of some datum, "something that is immediately given". Gödel does not deny that there is an ideal reality at the basis of mathematics. But he cannot hold that mathematical theories literally describe that ideal reality. Mathematics may rely on but does not *describe* an independent reality, and what it seems to describe, a universe of sets, does not exist independently of the mind. In fact, in Gödel's 'Platonism', mathematical objects, such as sets, may be seen as "mixture of subjective and objective elements":

> (19) "*The second alternative* [that mathematical objects are created out of something else] *is perfectly sufficient to prove the Platonistic point of view.* For they then exist in the same sense as physical objects since our ideas of these no doubt are a *mixture of objective and subjective elements.*" [Gödel s. d. a., box 20, from Gödel's hand on a text by Wang, my emphasis]

Such notes show that Gödel has abandoned the clear-cut Platonism defined by claim (i). In 1964, Gödel does not defend (i) but is only concerned with weaker claims, such as (ii) or (ii'). These claims, (ii) and (ii'), are sufficient for what Gödel calls a 'Platonistic view'. They are the same criteria that Gödel used to show that Brouwer's Intuitionism is on the side of 'Objectivism'. There seems to be no difference between 'Platonism' and 'Objectivism' in Gödel's terminology, except that the tradition in philosophy of mathematics makes it easier to apply to Intuitionism the second than the first.

Even though Gödel believes that mathematical objects are created out of something else, a datum that has an independent reality, he is ready to consider seriously any position that complies with the weak requirement that mathematical objects are not created by the *ego* out of nothing (with then also possibility (B), that mathematics is related to an unconscious reason). What Gödel denies is only that mathematical theories are entirely designed in a conscious activity, that we choose the properties of our objects in an act comparable to a decision. That is Gödel's Platonism. As quotations (2), (4), (7), (14) and (19) show, Gödel qualifies as 'Platonism', 'Objectivism', or 'Realism', positions complying simply with claim (ii), or (ii'), and not necessarily with (i). Thus one must be cautious when Gödel says of himself that he is a Platonist. He is not generally using the term in the sense of (i).

Gödel's requirements for the objectivity of mathematics are weak. Intuitionists such as Brouwer may comply with them. A Cognitivist, who would relate mathematics with properties of the brain, would also comply with them. A Sociologist, who would relate mathematics with social structures (arguing that the mathematics that we do depend on the education that we receive or the influence of such and such scientific institutions), would comply with claim (ii), since, though our mathematics may be accidental, their feature could not be said to be chosen by the *ego*.

Thus the strong claims in Gödel's epistemology do not come from his conception of the objectivity of mathematics. The difficulty is not in his 'Platonism' but comes from elsewhere. Let us recall the discussion with Brouwer. Gödel explicitly states that Brouwer's account "does not exclude 'Objectivism'" (see quotation (14)). Gödel's own position and, eventually, his rejection of intuitionism seem to depend on another claim, which is not directly related to the question of objectivity (at least, in the sense in which Gödel understands the word). Gödel believes that

(iii) Mathematics (or something in mathematics) is independent of the specific properties of the human mind.

This claim is not clearly formulated in Gödel's writings. However, it is clear (see for example quotations (15) and (16)) that Gödel does believe in the existence of rational beings that are not human and do not possess the same properties as the human mind. And these beings, such as God, should be able to do some mathematics. Therefore, there must be something in mathematics that is independent of the specific properties of the human mind. In fact, Gödel seems to have tried to investigate the possible mathematics of non-human minds, in particular minds that would not live in time or, at least, not in time as we know it.[23] Gödel is looking for a general theory of rationality, a theory that would apply to all rational beings:

(20) "While e.g. the theorems of cardinal numbers apply to *all* cardinal numbers or to the system of cardinal numbers in its whole extent, we know literally nothing that would apply to the total existence of rational beings (to all rational beings in all their phases of existence)." [Gödel s. d. a., box 8c, folder 117, item 040396, draft of a letter to Wang for [Wang 1974]]

I will in the two next paragraphs discuss this last claim (iii) with another example, Husserl.

8 Gödel and Husserl

Gödel most admired Husserl's essay of 1913, *Ideas pertaining to a Pure Phenomenology.* However, Gödel's relationship to Husserl is an ambiguous one, and it seems to evolve with time. I will give an example.

In this essay, of 1913, Husserl acknowledges that we have an intuition of mathematical objects. But he maintains that all the constituents, all the strata of this mathematical intuition, can be made conscious and then appears as 'life-experiences' *(Erlebnisse)* of the *ego*. In other words, the *ego* has in itself all that is necessary to create its objects, and it can analyse its acts and their material. If Husserl refuses to say that the *ego* creates its objects, since they existed for the *ego* before the *ego* started this analysis, the *ego* may at least re-enact their creation or trace their

23. In [Gödel s. d. b., VI, 431–432], Gödel wonders about the possibilities, which a non-Archimedean time would open.

constitution. In this measure (even though, in Husserl's terminology, the 'constitution' is not a 'creation'), transcendental phenomenology does not comply with claim (ii).

However, it is to be noted that Husserl's phenomenology complies with claim (iii). That comes from what Husserl (1913) calls the '*a priori* of the correlation' between the object (or, technically, the '*noema*') and our 'life-experiences'. Namely, Husserl claims that when we have the intuition of an object (when we see a red cube or when we perceive some concept), our mental acts or, more generally, our 'life-experiences' are *a priori* correlated to that object. Certain life-experiences are required for the intuition of that object, so that any other being who would perceive the same object would also have the same life-experiences. If God could perceive this cube that I see, he would have to see in the same way that I do and have the same life-experiences that I have [Husserl 1913, § 150]. Thus, it is possible to isolate in our internal experience a certain core that is necessary, that is required *a priori* for the intuition of the object and that, therefore, any kind of being must possess in order to perceive the same object. Contrary to his *Philosophy of Arithmetic* and to his latter *Experience and Judgment,* Husserl's essay of 1913 complies with Gödel's last requirement.

There is at least a fragment where Gödel acknowledges a disagreement with Husserl.

> (21) "Transc[endantal] *ego*: *ego* in so far as it takes a cognitive attitude and in it is guided by reason (with which it has contact)
>
> Husserl Ideas § 80 inconsistent: *ego*+reason; activity and passivity." [Gödel s. d. a., box 1c, folder 67] [24]

In the paragraph 80 of the essay of 1913, Husserl distinguishes, on one side, life-experiences that have the character of an act and in which the *ego* is present, and, on the other side, life-experiences, such as the *Hyle,* that only form the surroundings, the '*Milieu*' of mental acts, and the material from which a peculiar process (the '*noesis*') produces the object of our intuition. Husserl adds that such life-experiences as the Hyle "do not have the characteristic relation to the *ego* [...] but they also belong to the *ego*, they are his". Now this remark could indeed appear contradictory to Gödel. If some life-experiences do not have an apparent relation to the *ego* and do not *appear* as 'his', there is then no

24. On the margin of a letter from W. A. Howard. This item is already discussed in [Cassou-Noguès 2005].

reason to make them part of the *ego*. And, in fact, in Gödel's view, there is at the basis of mathematical knowledge some data which do not belong to the *ego* and cannot be analysed: they either come from another part of the mind, different from the *ego* (possibility (B), earlier on) or they are received passively from the outside and cannot themselves be constituted in the *ego*. Husserl, in Gödel's view, seems to include these data in the *ego*, at the price of an inconsistency. In reality, an introspective analysis of mathematical intuition, as the phenomenological analysis is, must remain incomplete. There is, at the end, a kind of opacity, a datum that cannot be analysed.

However, in this context, Gödel's lecture of 1961 appears as problematic. Gödel here refers to phenomenological reflection as a possible method for discovering new axioms. I quote two long texts that I will then discuss.

> (22) "In what manner, however, is it possible to extend our knowledge of [...] abstract concepts? [...] The procedure must thus consist, at least to a large extent, in a clarification of meaning that does not consist in giving definitions.
>
> Now in fact, there exists today the beginning of a science which claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl. Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts." [Gödel 1961, 383]

> (23) "If one considers the development of a child, one notices that it proceeds in two directions: it consists on the one hand in experimenting with the objects of the external world and with its [own] sensory and motor organs, on the other hand in coming to a better and better understanding of language, and that means—as soon as the child is beyond the most primitive designating [of objects]—of the basic concepts on which it rests. [...] Now one may view the whole development of empirical science as a systematic and conscious extension of what the child does when it develops in the first direction. [...and] one has examples where [...] a considerable further development takes place in the second direction. [...] Namely, it turns out that in the systematic establishment of the axioms of mathematics, new axioms, which do

> not follow by formal logic from those previously established, again and again become evident." [Gödel 1961, 383–384]

This second text may be related to Gödel's letter to C. Reid (see quotation (12)). Gödel then spoke of the 'conceptual outfit' that the child already possesses and on which all science seems to rest. Here, in quotation (23), Gödel distinguishes two directions: physical science extends the application of these concepts to the empirical reality; mathematics rather proceeds from a reflection on these concepts. The first text, quotation (22), relates the development of mathematics, the discovery of new axioms, to a phenomenological reflection. It is by turning our attention onto the activity involved in the intuition of a concept that we can find new axioms concerning that concept. Such a reflection is, in Husserl's phenomenology, an analysis of the constitution of the concept, an analysis that will eventually make clear all our 'life-experiences' and enable the *ego* to trace the constitution of the concept.

This text then frankly contradicts Gödel's argument for the objectivity of mathematics. For example, in quotation (6), Gödel argued that, if the *ego* had created its objects out of nothing, then it would be able to discover all their properties simply by analysing its acts ("by raising his creative activity from the subconscious to the conscious level"). And Gödel denied that this was possible. But, in quotation (22) and the remainder of the lecture of 1961, Gödel precisely refers to such an introspective analysis. He argues that such a reflexive analysis would lead to the discovery of axioms, and, eventually, give a non-mechanical procedure that would overcome the incompleteness results. Now the introspective analysis seems to enable one to obtain all the properties of the objects, so that it seems (if one applies Gödel's earlier criterion) that the *ego* could have created its objects out of nothing. In fact, Gödel's uncritical reference to Husserl seems to indicate that he now considers it possible to trace back the constitution of mathematical objects in the *ego*.

That would also contradict quotation (21) on Husserl's inconsistency. This fragment implied that the reflexive analysis in Husserl's phenomenology must be incomplete. Sooner or later, it must meet a datum that can no longer be analysed, either because the datum is simply received from the outside or because it is produced in a part of the mind to which the *ego* has no access. However, in the lecture of 1961, the reflexive procedure seems to lead to a complete analysis of mathematical concepts.

Thus, the lecture of 1961 contradicts other texts on the question of mathematical objectivity. It is difficult to give it a status. The notes from the unpublished papers may not always be dated. However, there is evidence that Gödel holds claims (ii), or (ii'), in 1964 and later on, at the time of his conversation with Wang. One could imagine a possible scenario with three parts: In 1946 (with "Russell's mathematical logic") and still in 1951, Gödel would have held a strong Platonism such as described in claim (i); his reading of Husserl in the mid-fifties would have led him to abandon that position and, for a time, around 1961, to adopt a strictly phenomenological attitude; however he would have come back around 1964 to a weak Platonism described by claims (ii) and (iii), and relying on an argument that he had devised earlier. I must admit that I do not know the motivations of these changes.

Conclusion

My aim, in this paper, has been to distinguish, using Gödel's texts, three different meanings for the objectivity of mathematics. There is first a strong claim:

> (i) "mathematics describes a non-sensual reality, which exists independently both of the acts and of the dispositions of the human mind."

Thus, our theories describe an ideal reality. Set theory describes a universe of sets that exists in itself. Putting aside Gödel's vocabulary, I think one could reserve the term "Platonism" for such a strong position, which Gödel has held, in 1946 in his paper on "Russell's mathematical logic", as in fact Russell himself.

Now, one can find a material in Gödel's writings to define two weaker meanings for the objectivity in mathematics.

> (ii') mathematical theories have not been invented "by the *ego*" (i. e. consciously) and "out of nothing".
>
> (iii) mathematics (or something in mathematics) is independent of the specific properties of the human being.

Claim (iii) concerns the non-anthropological character of mathematics. Though Gödel does not use the term in this context, claim (iii) can be considered as giving another meaning to the idea of 'objectivity'. Claims (ii) and (iii) characterise two different versions of the objectivity of mathematics. They are independent: Brouwer according to Gödel holds (ii) but not (iii), whereas Husserl could hold (iii) but not (ii). The conjunction of (ii) and (iii) does not imply (i). Gödel, in 1964, holds (ii) and (iii) but not (i). Thus, again putting aside Gödel's own terminology, claims (ii) and (iii) may be considered as two different ways to reach a non Platonistic objectivity in philosophy of mathematics.

The importance of these two claims (ii) and (iii) is that they clarify the meaning of "objectivity" in philosophy of mathematics. They would permit an analysis of positions in philosophy of mathematics, both traditional and contemporary. [25]

Claim (ii)	Yes	No
Claim (iii)		
Yes	Gödel in 1964	Husserl's phenomenology
No	Brouwer, according to Gödel	

Bibliography

Balaguer, Michael

1998 *Platonism and Anti-Platonism in Mathematics*, Oxford: Oxford University Press.

Cassou-Noguès, Pierre

2005 Gödel and the objective existence of mathematical objects, *History and philosophy of logic*, 26, 211–228.

2010 Virtual platonisms: Lautman and Gödel, in *Postanalytic and Metacontinental*, edited by Reynolds, J. et al, Continuum, 216–235.

Dedekind, Richard

1888 *Was sind und was sollen die Zahlen*, Braunschweig: Vieweg, english translation by W. W. Beman : "The nature and meaning of numbers", in *Essays on the Theory of Numbers*, New-York: Dover, 1963, 3–81.

25. In fact, my interest in these claims first came from the fact that they permit an analysis of the positions of philosophers coming from different traditions. In that perspective, an interesting example of Platonism, which would have been too long to discuss here, would be Lautman's Dialectical Platonism. Gödel has read some of Lautman's texts and, as Gödel, Lautman seems to hold (ii), (iii), but not (i) [Cassou-Noguès 2010].

Detlefsen, Michael

1998 Constructive existence claims, in *Philosophy of Mathematics Today*, edited by M. Schirm, Oxford: Clarendon Press, 307–335.

Giono, Jean

1961 *Noé*, Paris: Gallimard.

Gödel, Kurt

1951 Some basic theorems on the foundations of mathematics and their implications, in *Collected Works*, edited by S. Feferman, J. Dawson et al., Clarendon Press, vol. III, 304–324.

1961 The modern development of the foundations of mathematics in the light of philosophy, in *Collected Works*, edited by Feferman, S. & Dawson, J. et al, Oxford: Clarendon Press, vol. III, 374–387.

1964 What is Cantor's continuum problem?, in *Collected Works*, edited by S. Feferman, J. Dawson et al., Clarendon Press, vol. II, 254–270.

1972 On an extension of finitary mathematics, in *Collected Works*, edited by S. Feferman, J. Dawson et al., Clarendon Press, vol. II, 271–280.

1986–2003 *Collected Works*, vol. V, Oxford: Clarendon Press.

s. d. a. Papers from the Institute for Advanced Studies kept at the Library of Princeton University.

s. d. b. Philosophical notebooks, from Gödel's Papers, transcribed into German from Gabelsberger by Cheryl Dawson.

Husserl, Edmund

1913 *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Halle: Niemeyer.

1936 *Die Krisis der europäischen Wissenschaften*, Hamburg: Meiner, 1996.

Martin, Davis

2005 Gödel's conceptual realism, *Bulletin of Symbolic Logic*, 11, 207–235.

Parsons, Charles

1995 Platonism and mathematical intuition in Kurt Gödel's thought, *Bulletin of Symbolic Logic*, 1(1), 44–74.

Potter, Michael

2001 Was Gödel a Gödelian platonist?, *Philosophia Mathematica*, 9(3), 331–346.

Shapiro, Stewart

2007 The objectivity of mathematics, *Synthese*, 156(2), 337–381.

STEVENSON, ROBERT LOUIS
1892 *Across the Plains*, London: Chattus and Windus.

WANG, HAO
1974 *From Mathematics to Philosophy*, New York: Humanities Press.
1996 *A Logical Journey From Gödel to Philosophy*, Cambridge: Cambridge University Press.

WRIGHT, CRISPIN
1992 *Truth and Objectivity*, Cambridge (Mass.): Harvard University Press.